Constraints on Cosmic Quintessence and Quintessential Inflation

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Recently, attempts have been made to understand the apparent near coincidence of the present dark energy and matter energy in terms of a dynamical attractor-like solution for the evolution of a "quintessence" scalar field. In these models the field couples with the dominant constituent and only acts like a cosmological constant after the onset of the matter dominated epoch. A generic feature of such solutions, however, is the possibility of significant energy density in the scalar field during the radiation dominated epoch. This possibility is even greater if the quintessence field begins in a kinetic-dominated regime generated at the end of "quintessential inflation." As such, these models can affect, and therefore be constrained by, primordial nucleosynthesis and the epoch of photon decoupling. Here we analyze one popular form for the quintessence field (with and without a supergravity correction) and quantify constraints on the allowed initial conditions and parameters for the effective potential. We also deduce constraints on the epoch of matter creation at the end of quintessential inflation.

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I. INTRODUCTION

Recent observations [1,2] of Type Ia supernovae at intermediate redshift, along with complementary observational constraints at low and intermediate redshift, as well as the power spectrum of the cosmic microwave background all indicate [3] that the universe may be presently accelerating due to the influence of a dominant dark energy with a negative pressure. The simplest interpretation of this dark energy is a cosmological constant for which the equation of state is $\omega \equiv P/\rho = -1$. A second possibility is derived from the so-called "quintessence" models. In this case the dark energy is the result of a scalar field Q slowly evolving along an effective potential $V(Q)$. The equation of state is negative $-1 \leq w_Q \leq 0$, but not necessarily constant.

Introducing either of these paradigms, however, leads inevitably to two nagging questions. One is a fine tuning problem as to why the present dark energy is so small compared to the natural scales of high-energy physics.

The second is a cosmic coincidence problem as to why the universe has conspired to produce nearly equivalent energy content in matter and dark energy at the present time.

Attempts have been made [4–10] to reformulate this quandary by introducing specific forms of the quintessence effective potential whereby a tracker field Q evolves according to an attractor-like solution to the equations of motion. That is, for a wide variety of initial conditions, the solutions for Q and $\dot{Q}$ rapidly approach a common evolutionary track. The nice feature of these solutions is that they lead naturally to a cross over from an earlier radiation-dominated solution to one in which the universe is dominated by a small dark energy at late times. Another interesting possible feature is that such models might also naturally arise during matter creation at the end of an earlier "quintessential" inflationary epoch [11]. In this case, the Q field emerges in a kinetic-dominated regime at energy densities above the tracker solution.

It is not yet clear, however, [12,13] that these models have altogether solved the fine-tuning and cosmic-coincidence problems. Nevertheless, several such tracker fields have been proposed [10]. Although there is some difficulty in aligning quintessence models and string theory [14], the form for the effective potentials may at least be suggested by particle physics models with dynamical symmetry breaking, by nonperturbative effects [5], by generic kinetic terms "k-essence" in an effective action describing the moduli and massless degrees of freedom in string and supergravity theories [6–8], or by static and moving branes in a dilaton gravity background [15].

A general feature of all such solutions, however, is that at least the possibility exists for a significant contribution of the Q field to the total energy density during the radiation-dominated or photon decoupling epochs as well as the present matter-dominated epoch. The yields of primordial nucleosynthesis and the power spectrum of the CMB are strongly affected by the background energy density and universal expansion rate. Therefore, the possibility exists to utilize primordial nucleosynthe-

sis and the CMB power spectrum to constrain otherwise viable quintessence or k-essence models.

Observational constraints on such quintessence models have been of considerable recent interest. For example, in [16] the constraints on an M-theory motivated exponential form with a quadratic prefactor [17] for the quintessence effective potential were considered. In this model, the Q field closely follows the background field. In [18] a study was made of the effect on primordial nucleosynthesis of extended quintessence with nonminimal coupling. In such models the quintessence field couples with the scalar curvature. It can then act to slow the expansion and affect the yields of primordial nucleosynthesis. In the present work, we consider the power-law effective potential and its supergravity-corrected form as described below. We construct a detailed mapping of the allowed parameter space for these quintessence models. We also consider constraints these considerations place on the epoch of matter generation at the end of quintessential inflation.

II. QUINTESSENCE FIELD

A variety of quintessence [10] effective potentials or k-essence effective actions [6–8] can be found in the literature. In this paper, however, we will not consider k-essence effective actions as we have found that the attractor-like solution is limited to a prohibitively small range of parameters of the effective Lagrangian. We will, however, consider kinetic dominated quintessence models. Here, we concentrate on what is a frequently invoked form for the effective potential of the tracker field, i.e. an inverse power law such as originally analyzed by Ratra and Peebles [19],

$$V(Q) = M^{(4+\alpha)}Q^{-\alpha} \quad , \qquad (1)$$

where, M and α are parameters. The parameter M in these potentials is fixed by the condition that $\Omega_Q = 0.7$ at present. Therefore,

$$\rho_Q(0) = 0.7\rho_c(0) = 5.7h^2 \times 10^{-47} \text{ GeV}^4 \qquad (2)$$

and

$$M \approx \left(\rho_Q(0)Q^\alpha \right)^{1/(\alpha+4)} \quad . \qquad (3)$$

If Q is presently near the Planck mass and α is too small (say $\alpha \lesssim 2$), this implies a small value [5] for M which in a sense reintroduces the fine tuning problem.

We will also consider a modified form of $V(Q)$ as proposed by [20] based upon the condition that the quintessence fields be part of supergravity models. The rewriting of the effective potential in supergravity depends upon choices of the Kähler potential [13]. The flat

Kähler potential yields an extra factor of $exp\{3Q^2/2m_{pl}^2\}$ [20]. This comes about by imposing the condition that the expectation value of the superpotential vanishes. The Ratra potential thus becomes

$$V_{SUGRA}(Q) \rightarrow M^{(4+\alpha)}Q^{-\alpha} \times exp(3Q^2/2m_{pl}^2) \quad , \qquad (4)$$

where the exponential correction becomes largest near the present time as $Q \rightarrow m_{pl}$. This supergravity motivated effective potential is known as the SUGRA potential. The fact that this potential has a minimum for $Q = \sqrt{\alpha/3}m_{pl}$ changes the dynamics. It causes the present value of w_Q to evolve to a cosmological constant ($w_Q \approx -1$) much quicker than for the bare power-law potential [10].

The quintessence field Q obeys the equation of motion

$$\ddot{Q} + 3H\dot{Q} + dV/dQ = 0 \quad , \qquad (5)$$

where the Hubble parameter H is given from the solution to the Friedmann equation,

$$H^2 = \left(\frac{\dot{a}}{a} \right)^2 = \frac{1}{m_{pl}^2}(\rho_B + \rho_Q) \quad , \qquad (6)$$

where $m_{pl} = (8\pi G/3)^{-1/2} = 4.2 \times 10^{18}$ GeV, ρ_B is the energy density in background radiation and matter, and a is the cosmic scale factor.

For the simple inverse power-law potential, it can be shown [19] that the tracker solution maintains the condition

$$d^2V/dQ^2 = (9/2)(1 - \omega_Q^2)[(\alpha + 1)/\alpha]H^2 \quad . \qquad (7)$$

As the Q field evolves, its contribution to the energy density is given by

$$\rho_Q = \frac{\dot{Q}^2}{2} + V(Q) \quad , \qquad (8)$$

which may or may not be comparable to the energy density in radiation during the nucleosynthesis or photon-decoupling epoch depending upon the parameters and initial conditions.

The quintessence initial conditions are probably set in place near the inflation epoch. By the time of the big bang nucleosynthesis (BBN) epoch, many of the possible initial conditions will have already achieved the tracker solution. However, for initial conditions sufficiently removed from the tracker solution, it is quite possible that the tracker solution has not yet been achieved by the time of BBN. Such possibilities are illustrated schematically in Figure 1. These correspond to cases in which the initial energy density falls above (curve A), near (curve B) or below (curve C) the tracker solution.

III. NUCLEOSYNTHESIS CONSTRAINT

A. Quintessence and BBN

There are several paradigms of possible interest for constraint by primordial nucleosynthesis. These depend upon the initial values for the energy density in the Q field. In any of these cases, the energy density in ρ_Q can be constrained by the ratio ρ_Q/ρ_B during the BBN epoch at $0.01 \lesssim T \lesssim 1$ MeV, $10^8 \lesssim z \lesssim 10^{10}$ as shown in Figure 1.

If the initial conditions are sufficiently close to the tracker solution $\bar{\rho}_Q$ before nucleosynthesis, then $\rho_Q \sim \bar{\rho}_Q$ during nucleosynthesis and the tracker solution specifies the energy density in the Q field. This situation is similar to curve B in Figure 1. This may be the most likely scenario. Along the tracker solution, ρ_Q diminishes in a slightly different way than the radiation-dominant background energy density. For example, as long as $\rho_Q << \rho_B$, the Q-field decays as

$$\rho_Q \propto a^{-3(1+w_Q)} \quad , \tag{9}$$

with

$$w_Q = (\alpha w_B - 2)/(\alpha + 2) < w_B \quad . \tag{10}$$

The equation of state w_Q is only equal to the background equation of state w_B in the limit $\alpha \to \infty$. This is equivalent to the exponential potential. Nevertheless, the tracker solution does not deviate much from ρ_B, even at high redshift for most values of α considered here. Hence, one can characterize the nucleosynthesis results by the (nearly constant) ratio ρ_Q/ρ_B during the BBN epoch.

If the energy density in the tracker solution is close to the background energy density, the nucleosynthesis will be affected by the increased expansion rate from the increased total energy density. Such a situation occurs for large values of the power-law exponent α.

A second possibility is that the energy density ρ_Q could exceed the tracker solution and be comparable to or greater than the background energy density during primordial nucleosynthesis. This situation is something like curve A in Figure 1. The kinetic energy in the Q field dominates over the potential energy contribution to ρ_Q and $w_Q = +1$ so that the kinetic energy density diminishes as a^{-6}. In this case there could be a significant contribution from ρ_Q during nucleosynthesis as the Q field approaches the tracker solution. The strongest constraints on this case would arise when ρ_Q is comparable to the background energy density near the time of the weak-reaction freese out, while the later nuclear-reaction epoch might be unaffected. This case A is particularly interesting as this kinetic-dominated evolution could be generated by an earlier quintessential inflation epoch [11] as described below.

A final possibility might occur if the Q field approaches the tracker solution from below as indicated by curve C

in Figure 1. In this case, the tracker solution may be achieved after the the BBN epoch so that a small ρ_Q during BBN is easily guaranteed. In such models, however, the ultimate tracker curve might have a large energy density at a later epoch which could be constrained by the later CMB epoch as described below.

B. Light-Element Constraints

The primordial light-element abundances deduced from observations have been reviewed by a number of recent papers [21–24]. There are several outstanding uncertainties. For primordial helium there is an uncertainty due to the fact that the deduced abundances tend to reside in one of two possible values. Hence, for ^{4}He we adopt a rather conservative range:

$$0.226 \leq Y_p \leq 0.247.$$

For deuterium there is a similar possibility for either a high or low value. Here, however, we adopt the generally accepted [22,23] low values of

$$2.9 \times 10^{-5} \leq \text{D/H} \leq 4.0 \times 10^{-5}.$$

The primordial lithium abundance can be inferred from the surface abundances of old halo stars. There is, however, some uncertainty from the possibility that old halo stars may have gradually depleted their primordial lithium. Because of this we do not consider primordial lithium in the discussions here. The adoption of the low deuterium abundance forces the BBN primordial helium to be near its upper limit.

Adding energy density from the Q field tends to increase the universal expansion rate. Consequently, the weak reaction rates freeze out at a higher temperature T_w. This fixes the neutron to proton ratio ($n/p \approx \exp[(m_p - m_n)/T_w]$) at a larger value. Since most of the free neutrons are converted into ^{4}He, the primordial helium production is increased. Also, since the epoch of nuclear reactions is shortened, the efficiency of burning deuterium into helium is diminished and the deuterium abundance also increases. Hence, very little excess energy density from the Q field is allowed.

Figure 2 summarizes allowed values of ρ_Q/ρ_B at $T = 1$ MeV ($z \approx 10^{10}$) based upon the nucleosynthesis constraints. The region to the right of the curve labeled D/H corresponds to models in which the primordial deuterium constraint is satisfied, $D/H \leq 4.0 \times 10^{-5}$. The region below the line labeled Y_p corresponds to $Y_p \leq 0.247$. The hatched region summarizes allowed values for the energy density in the quintessence field during the nucleosynthesis epoch.

This figure is similar to that obtained in Freese et al. [25] based upon the light-element constraints available at

that time. They similarly considered dark energy densities which scale proportionally to the background radiation energy density. However, in their study a somewhat larger range of possible dark energies was allowed due to the larger uncertainties in the observed primordial abundances and the neutron half life at that time.

In the present work we deduce an absolute upper limit of 5.6% of the background radiation energy density allowed in the quintessence field. This maximum contribution is only allowed for $\eta_{10} \approx 4.75$ or $\Omega_b h^2 \approx 0.017$. A smaller contribution is allowed for other values of η_{10}. Indeed, this optimum η_{10} value is 4σ less than the value implied by the cosmic deuterium abundance [22,23] $\Omega_b h^2 = 0.020 \pm 0.001$ (1σ) $(\eta_{10} = 5.46 \pm 0.27)$. The most recent independent determinations of $\Omega_b h^2$ from high-resolution measurements of the power spectrum of fluctuations in the cosmic microwave background favor a value even higher. Both the BOOMERANG [26] and DASI [27] data sets imply $\Omega_b h^2 = 0.022^{+0.004}_{-0.003}$ (1σ) $(\eta_{10} = 6.00^{+1.10}_{-0.81})$. The deuterium and CMB constraints together demand that $\eta_{10} \geq 5.19$ which would limit the allowed contribution from the Q field to $\leq 2\%$ of the background energy density.

The most restrictive CMB constraint on $\Omega_b h^2$ derives from demanding a flat universe ($\Omega_{tot} = 1.0$), and marginalizing the likelihood function over all parameters with assumed Gaussian priors [26] based upon measurements of large-scale structure and Type Ia supernovae. This gives $\Omega_b h^2 = 0.023 \pm 0.003$ (1σ). If one adopts this as a most extreme case, then $\Omega_b h^2 \geq 0.020$. This would correspond to $\eta_{10} \geq 5.46$. From Figure 2 this would imply a much more stringent constraint that only about 0.1% of the background energy density could be contributed by the Q field. Of course, this is only a 1σ constraint, and it is questionable as to whether the upper limit to Y_p is well enough established to rule out a contribution to the energy density at the 0.1% level. For the remainder of this paper we will adopt the more conservative constraint of 5.6%. Nevertheless, it is of interest to explore how the quintessence parameters allowed by BBN might improve should the constraints from BBN ever be so tightly defined. Therefore, we will consider 0.1% as a conceivable limit that demonstrates the sensitivity to BBN. Even the most conservative 5.6% limit adopted here corresponds to only about half of the energy density allowed in [25] for 3 neutrino flavors.

IV. EQUATION OF STATE CONSTRAINT

Another constraint on allowed parameters for the quintessence field derives from the simple requirement that the Q field behaves like dark energy during the present matter-dominated epoch. Stated another way, the equation of state should be sufficiently negative, i.e. $w_Q \equiv P_Q/\rho_Q < 0$ by the present time.

In general, w_Q is a time-dependent quantity. The energy density and pressure in the Q field can be written $\rho_Q = \frac{1}{2}\dot{Q}^2 + V(Q)$ and $P_Q = \frac{1}{2}\dot{Q}^2 - V(Q)$. This gives,

$$w_Q = 1 - 2V(Q)/\rho_Q \quad , \tag{11}$$

where the time dependence derives from the evolution of $V(Q)$ and $\rho(Q)$. A comprehensive study of the observational constraints on the present value of w_Q has been recently summarized by Wang et al. [3]. We adopt the same constraints deduced in that paper. Mindful of systematic errors, they adopted a most conservative approach based upon progressively less reliable data sets. Using the most reliable current low-redshift and microwave background measurements, they deduce limits of $-1 < w_Q < -0.2$ at the 2σ level. Factoring in the constraint from Type Ia supernovae reduces the range for the equation of state to $-1 < w_Q < -0.4$. This range derives from a concordance analysis of models consistent with each observational constraint at the 2σ level or better. A combined maximum likelihood analysis suggests a smaller range of $-0.8 < w_Q < -0.6$ for quintessence models which follow the tracker solution, though $w_Q \approx -1$ is still allowed in models with nearly a constant dark energy. In what follows, we also invoke these same three possible limit ranges for the present value of w_Q.

As we shall see, these limits place a strong constraint on the bare Ratra power-law potential for almost any value of α. However, the SUGRA corrected form of $V(Q)$ is only slightly constrained.

V. CMB CONSTRAINT

A third constraint on the quintessence field arises from its effect on the epoch of photon decoupling. There are two effects to be considered.

A. Look-back effect

One is the effect of the Q field on the observed microwave background in the case where the energy density in the quintessence field is negligible during photon decoupling. This has been considered in [10]. In this case the effect of the dark energy is to modify the angular distance-redshift relation [28]. The existence of dark energy during the look-back time to the surface of last scattering shifts the acoustic peaks in the CMB power spectrum to smaller angular scales and larger l-values. The amplitude of the first acoustic peak in the power spectrum also increases, but not as much for quintessence models as for a true cosmological constant Λ. The basic features of the observed power spectrum [26] can be fit [10] with either of the quintessence potentials considered here. Indeed, dark energy is required to reproduce both

the observed power spectrum and the Type Ia supernova data. For our purposes, this look-back constraint is already satisfied by demanding that $\Omega_Q = 0.7$ at the present time through Eqs. (2) and (3).

B. Energy in Q field

There is, however, an additional possible effect of the Q field on the CMB which we also use as a constraint. If the energy density in the Q field is a significant fraction of the background energy during the epoch of photon decoupling, it can increase the expansion rate. Increasing the expansion rate can also push the l values for the acoustic peaks in the spectrum to larger values and increase their amplitude [28]. Such an effect of an increased expansion rate has been considered by a number of authors in various contexts [16,28,29]. For our purposes, we adopt the constraint deduced by [16] based upon the latest CMB sky maps of the Boomerang [26] and DASI [27] collaborations that the density in the Q field can not exceed $\Omega_Q \leq 0.39$ during the epoch of photon decoupling. This implies a maximum allowed contribution of the Q field during photon decoupling of $\rho_Q/\rho_B = \Omega_Q/(1 - \Omega_Q) \lesssim 0.64$. Note, however, that the Q field behaves differently near photon decoupling than during the BBN epoch (cf. Fig. 1). After the transition from a radiation-dominated to a matter-dominated universe, the Q field (now coupled to the matter field) can contribute a much larger fraction of the background energy density (e.g. Curve B of Fig. 1) than in the BBN epoch. Hence, the constraint on ρ_Q field from BBN must be considered separately from the constraint at the CMB epoch.

VI. QUINTESSENTIAL INFLATION

Another possible constraint arises if the kinetic term dominates at an early stage (e.g. Curve A of Fig. 1). In this case $\rho_Q \approx \dot{Q}^2/2$ and ρ_Q decreases with scale factor as a^{-6}. At very early times this kinetic regime can be produced by so-called "quintessential inflation" [11]. In this paradigm entropy in the matter fields comes from gravitational particle production at the end of inflation. The universe is presumed to exit from inflation directly into a kinetic-dominated quintessence regime during which the matter is generated. An unavoidable consequence of this process, however, is the generation of gravitational waves along with matter and the quanta of the quintessence field [11,30–33] at the end of inflation.

A. Energy in Quanta and Gravity Waves

Particle creation at the end of inflation has been studied by the standard methods of quantum field theory in curved space-time [33]. The energy density in created particles is just [11,31–33]

$$\rho_B \approx \frac{1}{128} N_s H_1^4 \left(\frac{z+1}{z_1+1} \right)^4 \left(\frac{g_1}{g_{eff}(z)} \right)^{1/3} \ , \qquad (12)$$

where H_1 and z_1 are the expansion factor and redshift at the matter thermalization epoch at the end of quintessential inflation, respectively. The factor of 128 in this expression comes from the explicit integration of the particle creation terms.

When the gravitons and quanta of the Q field are formed at the end of inflation, one expects [11] the energy density in gravity waves to be twice the energy density in the Q-field quanta (because there are two graviton polarization states). In this paradigm then, wherever we have deduced a constraint on ρ_Q, it should be taken as the sum of three different contributions. One is the dark energy from the vacuum expectation value $\langle \rho_Q \rangle$ of the Q field; a second is the contribution from the fluctuating part $\rho_{\delta Q}$ of the Q field; and a third is from the energy density ρ_{GW} in relic gravity waves. Thus, we have

$$\rho_Q \to (\langle \rho_Q \rangle + \rho_{\delta Q} + \rho_{GW}) \ . \qquad (13)$$

For the cases of interest, the energy density in gravity waves and quanta scales like radiation after inflation, $\rho_{GW} + \rho_{\delta Q} \propto a^{-4}$, while the quintessence field vacuum expectation value evolves according to Eq. (9) which gives $\langle \rho_Q \rangle \propto a^{-6}$ during the kinetic dominated epoch. This epoch following inflation lasts until the energy in the Q field falls below the energy in background radiation and matter, $\rho_Q \leq \rho_B$.

Thus, for the kinetic dominated initial conditions (curve A) in particular, gravity waves could be an important contributor to the excess energy density during nucleosynthesis. The relative contribution of gravity waves and quintessence quanta compared to the background matter fields is just given by the relative number of degrees of freedom. At the end of inflation, the relative fraction of energy density in quanta and gravity waves is given by [11]

$$(\rho_{\delta Q} + \rho_{GW})/\rho_B = 3/N_s \ , \qquad (14)$$

where N_s is the number of ordinary scalar fields at the end of inflation. In the minimal supersymmetric model $N_s = 104$. Propagating this ratio to the time of nucleosynthesis requires another factor of $(g_n/g_1)^{1/3}$ where $g_n = 10.75$ counts the number of effective relativistic degrees of freedom just before electron-positron annihilation, and g_1 counts the number of degrees of freedom during matter thermalization after the end of inflation.

In the minimal standard model $g_1 = 106.75$, but in supergravity models this increases $\sim 10^3$.

Combining these factors we have

$$(\rho_{\delta Q} + \rho_{GW})/\rho_B \leq 0.014 \quad , \tag{15}$$

during nucleosynthesis. Hence, in this paradigm, the allowed values of ρ_Q/ρ_B consistent with nucleosynthesis could be reduced from a maximum of 0.056 to 0.042, further tightening the constraints deduced here.

B. Gravity-Wave Spectrum

There has been considerable recent interest [30–32] in the spectrum of gravity waves produced in the quintessential inflation paradigm. One might expect that the *COBE* constraints on the spectrum also lead to constraints on the Q field. Here we consider this possibility, but conclude that no significant constraint on the initial ρ_Q or effective potential is derived from the gravity wave spectrum. On the other hand, the BBN and CMB gravity-wave constraints discussed here can be used to provide useful constraints on the quintessential inflation epoch as we now describe. Our argument is as follows: The logarithmic gravity-wave energy spectrum observed at the present time can be defined in terms of a differential closure fraction,

$$\Omega_{GW}(\nu) \equiv \frac{1}{\rho_c} \frac{d\rho_{GW}}{d\ln\nu} \quad , \tag{16}$$

where the ρ_{GW} is the present energy density in relic gravitons and $\rho_c(0) = 3H_0^2/8\pi G = H_0^2 m_{pl}^2$ is the critical density. This spectrum has been derived in several recent papers [30–32]. It is characterized by spikes at low and high frequency. The most stringent constraint at present derives from the *COBE* limit on the tensor component of the CMB power spectrum at low multipoles. There is also a weak constraint from the binary pulsar [31,32] and an integral constraint from nucleosynthesis as mentioned above.

For our purposes, the only possible new constraint comes from the *COBE* limits on the tensor component of the CMB power spectrum. The soft branch in the gravity-wave spectrum lies in the frequency range between the present horizon $\nu_0 = 1.1 \times 10^{-18}\Omega_M^{1/2}h$ Hz and the decoupling frequency $\nu_{dec}(0) = 1.65 \times 10^{-16}\Omega_M^{1/2}h$ Hz, where we adopt $\Omega_M = 0.3$ for the present matter closure fraction. The constraint on the spectrum can be written [32],

$$\Omega_{GW}(\nu) = \Omega_\gamma \frac{81}{(16\pi)^2}\left(\frac{g_{dec}}{g_1}\right)^{1/3}\left(\frac{H_1}{m_{pl}}\right)^2 \times \left(\frac{\nu_{dec}}{\nu}\right)^2$$
$$\times \ln^2\left(\frac{\nu_r}{\nu_1}\right) \leq 6.9h^{-2} \times 10^{-11} \quad , \tag{17}$$

where, $\Omega_\gamma = 2.6 \times 10^{-5}h^2$ is the present closure fraction in photons. The number of relativistic degrees of freedom at decoupling is $g_{dec} = 3.36$. As noted previously, g_1 is the number of relativistic degrees of freedom after matter thermalization. In the minimal standard model is $g_1 = 106.75$. The quantity H_1 is the expansion rate at the end of inflation. In quintessential inflation it is simply related to the kinetic energy ρ_Q after inflation,

$$\rho_Q(z_1) = H_1^2 m_{pl}^2 \quad . \tag{18}$$

The logarithmic term in Eq. (17) involves present values of the frequency $\nu_r(0)$ characteristic of the start of radiation domination ($\rho_B = \rho_Q$ at $z = z_r$), and the frequency characteristic of matter thermalization at the end of inflation $\nu_1(0)$. The quantity ν_1 is just,

$$\nu_1(0) = \frac{H_1}{c}\left(\frac{z_0 + 1}{z_1 + 1}\right) \quad . \tag{19}$$

Similarly,

$$\nu_r(0) = \frac{H_r}{c}\left(\frac{z_0 + 1}{z_r + 1}\right) = \frac{H_1}{c}\left(\frac{z_r + 1}{z_1 + 1}\right)^3\left(\frac{z_0 + 1}{z_r + 1}\right) \quad , \tag{20}$$

so that,

$$\frac{\nu_r(0)}{\nu_1(0)} = \left(\frac{z_r + 1}{z_1 + 1}\right)^2 \quad . \tag{21}$$

The identity $\rho_B = \rho_Q$ at $z = z_r$ then gives,

$$\frac{\nu_r}{\nu_1} = \frac{N_s}{128}\left(\frac{H_1}{m_{pl}}\right)^2\left(\frac{g_1}{g_r}\right)^{1/3} \quad , \tag{22}$$

where for the cases of interest $g_r = 10.75$.

Collecting these terms, we can then use Eq. (17) to deduce a constraint on the expansion factor H_1 at the end of inflation

$$H_1^2 < 1.4 \times 10^{-11}m_{pl}^2 \quad . \tag{23}$$

For kinetic dominated models, $\rho_Q(z_1)$ at the end of inflation is simply related to the energy density ρ_Q at z,

$$\rho_Q(z) = \rho_Q(z_1)\left(\frac{z + 1}{z_1 + 1}\right)^6 \quad . \tag{24}$$

Similarly the background matter energy density scales as

$$\rho_B(z) = \rho_B(z_1)\left(\frac{g_1}{g_{eff}(z)}\right)^{1/3}\left(\frac{z + 1}{z_1 + 1}\right)^4 \quad . \tag{25}$$

Considering the present energy density in photons and neutrinos, we can find a relation between H_1 and z_1:

$$\rho_{\gamma 0} + \rho_{\nu 0} = 1.1 \times 10^{-125}m_{pl}^4$$
$$= \frac{N_s}{128}H_1^4\left(\frac{g_1}{g_{dec}}\right)^{1/3}(z_1 + 1)^{-4}, \tag{26}$$

We then deduce from this and Eq. (23) that $z_1 < 8.4 \times 10^{25}$. These equations then imply that there is only a lower limit on ρ_Q/ρ_B given by the constraint on H_1. Combining Eqs. (12), (24), and (25), we have

$$\rho_Q(z)/\rho_B(z) = \frac{128 m_{pl}^2}{N_s H_1^2}\left(\frac{g_1}{g_{eff}(z)}\right)^{-1/3}\left(\frac{z+1}{z_1+1}\right)^2 \; . \tag{27}$$

At our initial epoch $z = 10^{12}$, we then deduce that , $\rho_Q(z)/\rho_B(z) \gtrsim 5.6 \times 10^{-18}$. This limit is not particularly useful because ρ_Q field must exceed ρ_B at z_1 in order for the gravitational particle production paradigm to work. The implication is then that all initial conditions in which the kinetic term dominates over the background energy at z_1 are allowed in the quintessential inflation scenario. Hence, we conclude that the gravity-wave spectrum does not presently constrain the initial ρ_Q or $V(Q)$ in the quintessential inflation model.

Before leaving this discussion on quintessential inflation, however, we remark that the limits on $\rho_Q/\rho_B \leq 560$ derived from the BBN constraints discussed below, can be used to place a lower limit on the expansion rate at the end of inflation in this model. This in turn can be used to deduce a lower limit on the redshift for the end of quintessential inflation from Eq. (26). Thus, we have

$$2.2 \times 10^{-21} < \frac{H_1^2}{m_{pl}^2} < 1.45 \times 10^{-11} \; , \tag{28}$$

and in the minimal supersymmetric model

$$1.0 \times 10^{21} < z_1 < 8.4 \times 10^{25} \; . \tag{29}$$

This nontrivial constraint then implies that the kinetic driven quintessential inflation must end at an energy scale somewhere between about 10^8 and 10^{13} GeV, well below the Planck scale. By similar reasoning one can apply this argument to the gravity-wave spectrum from normal inflation as given in [32]. We deduce an upper limit to the epoch of matter thermalization of $H_1 \leq 3.1 \times 10^{-10} m_{pl}^2$ which implies $z_1 \leq 7.3 \times 10^{28}[g(z_1)/3.36]^{1/12}$. In this case there is no lower limit from BBN as there is no Q field present after inflation.

VII. RESULTS AND DISCUSSION

The equations of motion [Eqs. (5) and (6)] were evolved for a variety of initial Q field strengths and power-law parameters α. As initial conditions, the quintessence field was assumed to begin with equipartition, i.e. $\dot{Q}^2/2 = V(Q)$. This implies $w_Q = 0$ initially. This seems a natural and not particularly restrictive choice, since w_Q quickly evolves toward the kinetic ($w_Q = +1$) or the tracker solution [Eq. (10)] depending upon whether one begins above or below the tracker curve.

Nevertheless, this initial condition does not encompass all possible cases. For example, one might imagine a somewhat strange initial condition in which Q begins at a very large value such that $V(Q) << \rho_c(0)$ while $\rho_Q = \dot{Q}^2/2$ is arbitrarily large. In this case, ρ_Q will quickly decay away as a^{-6} before the present epoch and insufficient dark energy will be present at the current epoch. Hence, it is possible that even though one starts with a large ρ_Q, one may not satisfy the present equation of state constraint. We do not consider this as a likely case, however. The natural initial values for the quintessence field are $Q \leq m_{pl}$, so that sufficient dark energy should always be present for this potential choice. Besides, postulating an initially small value for Q reintroduces the small Λ problem which is what these models were invented to avoid.

Constraints on α and the initial value for the Q-field energy density $\rho_Q(z)/\rho_B(z)$ at $z = 10^{12}$ were deduced numerically. These are summarized in Figure 3 for both: (a) the bare Ratra power-law potential; and (b) its SUGRA corrected form. We present the quantity ρ_Q/ρ_B as a more intuitive measure of the relative amount of Q-field energy density than to just give ρ_Q. This quantity can be easily converted to $\Omega_Q = (\rho_Q/\rho_B)/(1 + \rho_Q/\rho_B)$. For purposes of illustration, we have arbitrarily specified initial conditions at $z = 10^{12}$, corresponding to $T \sim 10^{12}$ K, roughly just after the time of the QCD epoch. At any time the energy density $\rho_{rel}(z)$ in relativistic particles is just

$$\rho_{rel}(z) = \rho_{\gamma 0}\left(\frac{3.36}{g_{eff}(z)}\right)^{1/3}(z+1)^4 \; , \tag{30}$$

where $\rho_{\gamma 0} = 2.0 \times 10^{-51}$ GeV^4, is the present energy density in microwave background photons, and we take $g_{eff}(z) = 10.75$ between $z = 10^{12}$ and the beginning of BBN just before electron-positron annihilation ($z \approx 10^{10}$).

The envelope of models which obtain a tracker solution by the epoch of nucleosynthesis are indicated by upper and lower curved lines in Figures 3a and 3b. The general features of these constraints are as follows. If the initial energy density in the Q field is too large, the tracker solution is not reached by the time of BBN. The Q-field energy density can then significantly exceed the background energy during nucleosynthesis. This situation corresponds to the excluded regions on the top of Figures 3a and 3b.

All solutions consistent with the primordial nucleosynthesis constraints are also consistent with our adopted CMB Q-energy constraint as also shown at the top of Figures 3a and 3b. Similarly, if the initial energy in the Q field is too small, the universe does not become dark-energy dominated by the present time. This corresponds to the excluded (no Λ) region at the bottom of the figures.

The excluded regions at the top and bottom of figures 3a and 3b can be easily understood analytically. For example, the excluded (no Λ) region at the bottom of these figures reflects the fact that if ρ_Q is initially below the value presently required by $\Omega_\Lambda = 0.7$ [cf. Eq. (2)] it can not evolve toward a larger value. Hence, $\rho_Q/\rho_B < 2.8 \times 10^{-44}$ is ruled out for $h = 0.7$. Similarly, the "Excluded by BBN" region comes from requiring that $\rho_Q(z_{BBN})/\rho_B(z_{BBN}) < 0.056$. For this constraint we are only considering cases in which the Q field is approaching the tracker solution from above during nucleosynthesis. Hence, it is in the kinetic regime in which $\rho_Q \propto z^6$ while the background scales as z^4. Thus, we have $\rho_Q(z = 10^{12})/\rho_B(z = 10^{12}) > 0.056(10^{12}/10^{10})^{6-4} = 560$ is excluded. By similar reasoning, the "Excluded by CMB" region is given by $\rho_Q(z_{CMB})/\rho_B(z_{CMB}) > 0.64$, or $\rho_Q(z = 10^{12})/\rho_B(z = 10^{12}) > 0.64(10.75/3.36)^{1/3}(10^{12}/10^3)^{6-4} = 9.4 \times 10^{17}$.

For the bare Ratra power-law potential (Fig. 3a) the main constraint is simply the requirement that the equation of state be sufficiently negative by the present time. The sensitivity of the allowed power-law exponent to the equation of state is indicated by the w_Q = -0.2, -0.4, and -0.6 lines on Figure 3a. In the present Q-dominated epoch, Eq. (10) is no longer valid even in the tracker regime. Thus, the lines of constant w_Q must be evaluated numerically using Eq. (11). The slight slope to these curves comes from the fact that $V(Q)/\rho(Q)$ has not yet reached unity, i.e. there is still some small kinetic contribution to $\rho(Q)$ and the amount of kinetic contribution depends upon α.

For the bare Ratra power-law potential, tracker solutions with $\alpha \lesssim 20$ are allowed if $w_Q \leq -0.2$. The allowed values for α reduce to < 9 and < 2 if the more stringent -0.4 and -0.6 constraints are adopted. If α is too small, say $\alpha \leq 2$, then the potential parameter M becomes a very small fraction of the Planck mass [cf. Eq. (3)], and the fine tuning problem is reintroduced.

For models in which the tracker solution is obtained by the time of BBN, nucleosynthesis only limits the potential parameters if the most conservative equation of state limit ($w_Q < -0.2$) and most stringent nucleosynthesis constraint ($\rho_Q/\rho_B < 0.1\%$) are adapted. On the other hand, independently of the equation of state constraint, nucleosynthesis limits a large family of possible kinetic-dominated solutions in which the Q-field energy density exceeds the background energy prior to and during the nucleosynthesis epoch. Such models are excluded even though they provide sufficient dark energy in the present epoch.

For the SUGRA-corrected Q fields (Fig. 3b), the constraint from w_Q is greatly relaxed. In fact, w_Q is sufficiently negative ($w_Q < -0.6$) for all $\alpha < 10^4$. The reason is that all tracker solutions have $w_Q \approx -1$. This is because w_Q decays much faster toward -1 for the SUGRA potential. Also, the potential has a finite minimum which is equal to the present dark-energy density. The Q field quickly evolves to near the potential minimum and has negligible kinetic energy by the present time. Any potential which becomes flat at late times gives $w_Q \approx -1$ and the dark energy looks like a cosmological constant Λ. All SUGRA models which achieve the tracker solution also have a small ρ_Q during primordial nucleosynthesis. Hence, there is no constraint from nucleosynthesis except for those kinetic-dominated models in which the Q field is still far above the tracker solution during the nucleosynthesis epoch.

We do note, however, that if a lower limit of $w_Q > -0.8$ is adopted for tracker solutions from [9], then only a power law with $\alpha \gtrsim 30$ is allowed as indicated on Figure 3b. This makes the SUGRA potential the preferred candidate for quintessence. The large α implies values of M in Eq. (3) close to the Planck mass, thus avoiding any fine tuning problem. However, this quintessence potential is constrainable by BBN should the light-element constraints become sufficiently precise to limit ρ_Q at the 0.1% level.

VIII. CONCLUSIONS

We conclude that for both the bare Rata inverse power-law potential and its SUGRA-corrected form, the main constraints for models which achieve the tracker solution by the nucleosynthesis epoch is from the requirement that the equation of state parameter becomes sufficiently negative by the present epoch. The main constraint from nucleosynthesis is for models in which the Q field has not yet obtained the tracker solution by the time of nucleosynthesis. The SUGRA-corrected potential is the least constrained and avoids the fine tuning problem for M. Therefore, it may be the preferred candidate for the quintessence field, although BBN may eventually limit this possibility. We also note that the constraints considered here provide useful constraints on the regime of matter creation at the end of quintessential inflation.

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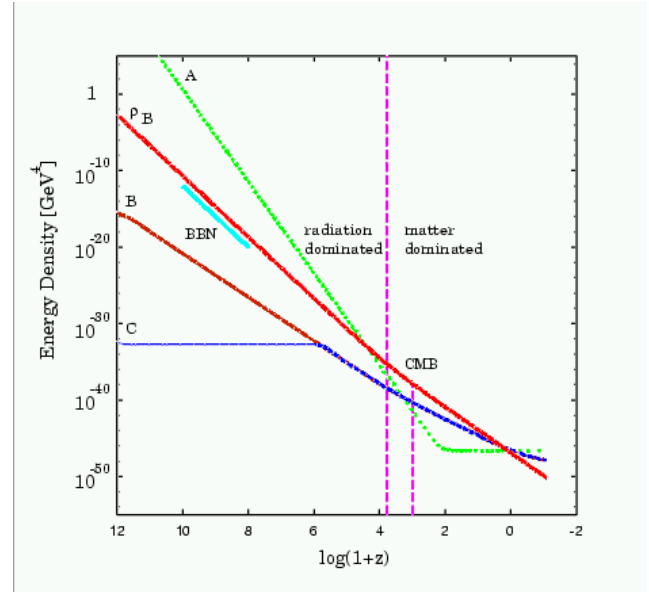FIG. 1. Examples of the evolution of energy density in ρ_Q and the background fields ρ_B as a function of redshift for an inverse power-law effective potential with $\alpha = 5$. The BBN epoch is indicated by a short line segment. The location of the transition from radiation to matter domination (for $\Omega_M = 0.3$), and the CMB epoch, are also indicated. Curves A and C illustrate cases in which the Q field does not achieve the tracker solution until well after the BBN epoch. Curve A is excluded, curve C is not. Curve B is an example of a tracker solution which is allowed by the nucleosynthesis constraints.
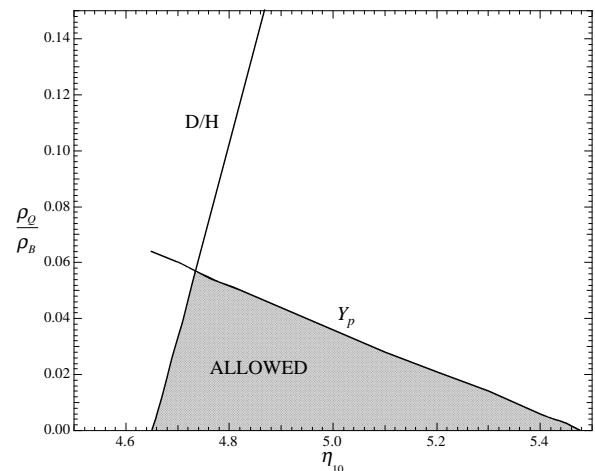
FIG. 2. Constraints on the ratio of the energy density in the quintessence field to the background energy density ρ_Q/ρ_B (at $T = 1$ MeV) from the primordial abundances as indicated. The allowed region corresponds to $Y_p \leq 0.247$ and D/H $\leq 4.0 \times 10^{-5}$.
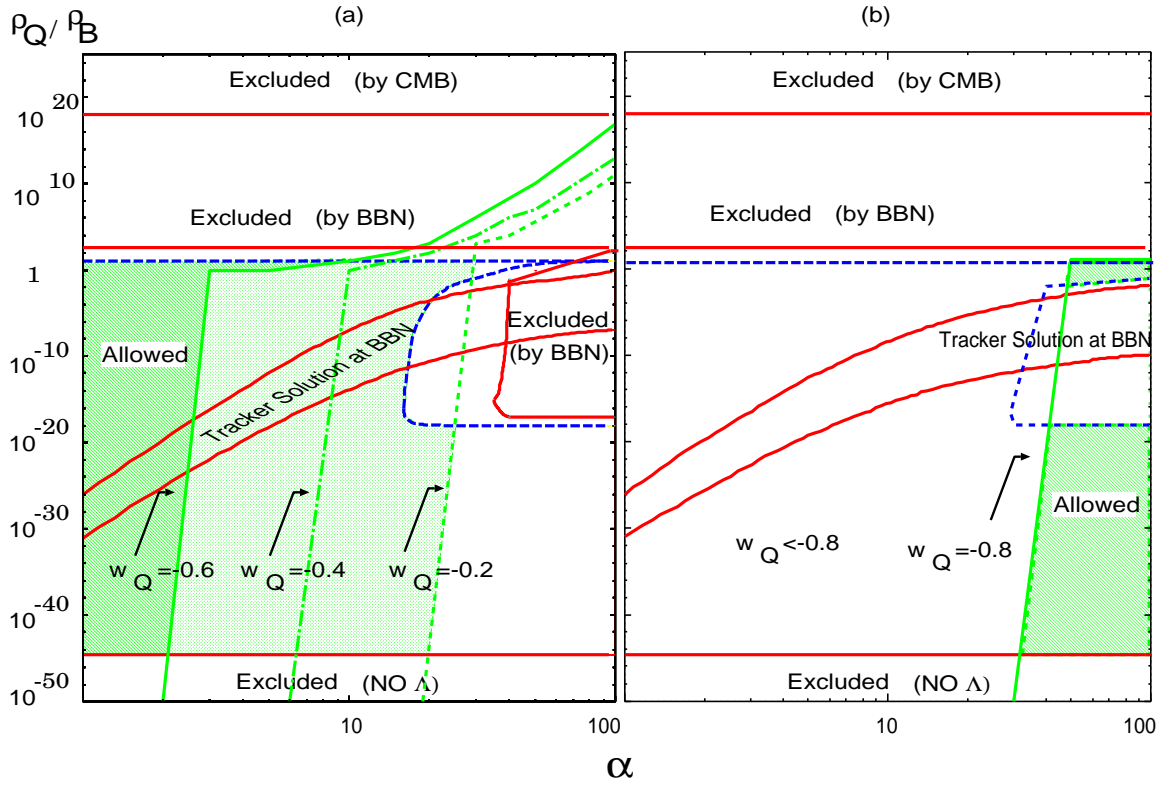
FIG. 3. Contours of allowed values for α and initial ρ_Q/ρ_B (at $z = 10^{12}$) from various constraints as indicated for (a) the bare power-law potential, and (b) the SUGRA corrected potential. Models in which the tracker solution is obtained by the BBN epoch are indicated by the upper and lower curves. Values of α to the right of the lines labeled $w_Q = -0.6, -0.4, -0.2$ on (a) are excluded by the requirement that the present equation of state be sufficiently negative. The BBN constraint for a maximum energy density in the Q field of 0.1% (dotted line) and 5.6% (solid line) are also indicated. For the SUGRA potential (b) all tracker solutions to the right of the region labeled $w_Q = -.8$ are allowed.

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